Multiscaling of noise-induced parametric instability

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We describe the statistical properties of growth rates of a linear oscillator driven by a parametric noise. We show that in general the fluctuations of local Lyapunov exponents are non-Gaussian and demonstrate multiscaling. Analytical calculations of the generalized Lyapunov exponents are complemented with approximative and numerical results; this allows us to identify the parameter range where the deviations from the Gaussian statistics become important.

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I. INTRODUCTION

In spite of recent progress in the studies of noise-induced dynamics, many simple models elude complete analytical solution. In this paper we investigate the dimensionless random-driven linear oscillator model

$$\ddot{x}+[E+\xi(t)]x=0, \quad \langle\xi(t)\rangle=0,$$

$$\langle\xi(t)\xi(t')\rangle=2\sigma^2\delta(t-t'), \tag{1}$$

with the Gaussian white noise $\xi(t)$. This simple model, which is not exactly solvable in the statistical sense, finds applications in various fields of physics.

(1) In the theory of Anderson localization one interprets t as a spatial coordinate; then Eq. (1) is the one-dimensional stationary Schrödinger equation for a single particle in a δ-correlated potential ξ. The parameter E is the energy eigenvalue; it can be either positive or negative, where negative values correspond to the band gap. Although, strictly speaking, one should consider Eq. (1) with two boundary conditions $|x|\to 0$ at $t\to\pm\infty$, the usual approach is to treat Eq. (1) as an initial-value problem and to look at the growth rate of the variable x as $t\to\infty$ (see [1,2,16] for details). The growth rate gives the localization length, and fluctuations of $x(t)$ are important for the description of conductance fluctuations in finite samples.

(2) In the theory of parametric resonance one assumes the parameter E to be positive and interprets it as the square of the oscillator frequency, while ξ describes frequency fluctuations. The oscillations $x(t)$ grow due to the noisy pump and the growth rate of different moments is of major interest. Usually, the oscillator has linear damping described by the term $2\gamma\dot{x}$; such an equation can be transformed to Eq. (1) by virtue of the transformation $x\to e^{-\gamma t}y,\ \ E\to E-\gamma^2$.

(3) In Refs. [3–5] a geometrical approach to calculation of the largest Lyapunov exponent in high-dimensional Hamiltonian systems was suggested. This approach leads to Eq. (1), where $x(t)$ is the amplitude of a small perturbation, E is the mean curvature of the potential energy (it can be of either sign), and ξ describes chaotic fluctuations of the curvature. The growth rate of $x(t)$ thus gives an estimate of the Lyapunov exponent of Hamiltonian chaos.

In all applications the growth rate of the oscillations $x(t)$ is of major interest. When this rate describes the growth of small perturbations, as in the last example above, it is understood that nearby trajectories evolve with the same realization of the noise (see [6] for a detailed discussion of this problem). This is consistent with the ususal definition of the Lyapunov exponent, which also gives the correct interpretation as the inverse of the localization length.

Generally, one cannot characterize the growth rate with a single number, so one speaks of multiscaling. This property can be characterized in a twofold way. On one hand it is possible to characterize the fluctuations of the exponential growth with the local (finite-time) Lyapunov exponent introduced as $\lambda(t)=(2t)^{-1}\ln(x^2+\dot{x}^2)$. Then one describes multiscaling in terms of the corresponding probability density $P(\lambda;t)$. Recently it has been shown [7], that for values of E close to zero the distribution of $\lambda(t)$ deviates from the Gaussian form, leading to nonvanishing higher cumulants. Thus the usual Lyapunov exponent alone is not sufficient for statistical characterization of $x(t)$ for large times. This is why we treat the problem with the help of generalized Lyapunov exponents, corresponding to the growth rates of different moments of the field x. In the presence of multiscaling these growth rates are different, which gives a complementary characterization. Our main goal in this paper is to define the range of parameters E,σ where multiscaling is essential and to relate the asymptotical scaling of the generalized exponents to the form of the tails of the density $P(\lambda;t)$.

This paper is organized in the following way. In the next section we introduce the generalized Lyapunov exponents and perform a time rescaling, which leaves a noise renormalized frequency as the sole relevant parameter. We recall some known properties of the Lyapunov exponent of the random oscillator, which we supplement with results for negative frequency E. The non-Gaussian properties of the distribution of the local Lyapunov exponent are treated in Sec. IV, using numerical results and some analytical estimates.

II. GENERALIZED LYAPUNOV EXPONENTS

A. Definition of Lyapunov exponents

We start with the definition of quantities that characterize the growth of oscillations in our basic model.

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The fluctuations in Eq. (1) lead to an exponential (on average) growth of the amplitude: $A=\sqrt{x^2+\dot{x}^2}\sim\exp(\Lambda t)$. Because of the similarity to linearized equations for the growth of small perturbations in chaotic systems, the exponent Λ is called the Lyapunov exponent (LE). The local (finite-time) LE is defined as

$$\lambda(t)=\frac{1}{t}\ln A(t); \tag{2}$$

it converges to the LE as time tends to infinity and is a self-averaged quantity:

$$\lambda(t)\underset{t\to\infty}{\to}\Lambda=\left\langle\frac{d}{dt}\ln A\right\rangle.$$

Due to fluctuations of $\lambda(t)$ the growth rate of the moments of A generally differs from Λ. It is possible to characterize these fluctuations with the help of the generalized Lyapunov exponents [8], defined as the growth rates of the moments of the amplitude:

$$L(q)=\frac{1}{q}\lim_{t\to\infty}\frac{1}{t}\ln\langle A^q(t)\rangle. \tag{3}$$

This definition includes the usual LE as a special case: $\Lambda=\lim_{q\to 0}L(q)$. Numerically, Λ is easier to calculate, because $\lambda(t)$ becomes a nonrandom quantity for large t [2]. Generally, all $L(q)$ are different and are necessary to characterize the growth of oscillations, as is discussed below in Sec. II C.

B. Analytic expressions for LEs

Remarkably, generalized LEs for $q=0,2,4,6,\ldots$ can be found analytically (see, e.g., [1,15]).

In the case $q=0$ we have the usual LE, which can be calculated as follows. With the ansatz $y=\dot{x}/x=d\ln x/dt$ one reduces Eq. (1) to the first-order nonlinear Langevin-type equation

$$\dot{y}=-y^2-\xi(t)-E.$$

Here y, after reaching $-\infty$, is reinjected at $+\infty$, which corresponds to a zero crossing of $x(t)$. The next step is the application of the Fokker-Planck theory: for the distribution of y one can write the Fokker-Planck equation and find its stationary solution (which, of course, is a solution with a constant probability flow):

$$\varrho(y)=\frac{1}{\sqrt{\pi}\sigma^{4/3}}\frac{e^{-y^3/3\sigma^2-Ey/\sigma^2}\int_{-\infty}^{y}e^{x^3/3\sigma^2+Ex/\sigma^2}dx}{\int_0^\infty x^{-1/2}e^{-x^3/12-Ex/\sigma^{4/3}}dx}.$$

Averaging $\langle y\rangle$ using this solution yields the following expression for the LE:

$$\Lambda=\langle y\rangle=\frac{\sigma^{2/3}}{2}\frac{\int_0^\infty x^{1/2}e^{-x^3/12-Ex/\sigma^{4/3}}dx}{\int_0^\infty x^{-1/2}e^{-x^3/12-Ex/\sigma^{4/3}}dx}. \tag{4}$$

For $q=2,4,6,\ldots$ another analytical approach can be used. Because Eq. (1) is a linear stochastic equation, the evolution of the moments of order q of the type $\langle x^{q-k}\dot{x}^k\rangle$ leads to a closed linear system of equations. The latter can be derived as follows (another way to derive this system is presented in [7]). Consider the temporal derivative of the moment expressed in terms of Eq. (1):

$$\frac{d}{dt}\langle x^{q-k}\dot{x}^k\rangle=(q-k)\langle x^{q-k-1}\dot{x}^{k+1}\rangle-kE\langle x^{q-k+1}\dot{x}^{k-1}\rangle-k\langle\xi(t)x^{q-k+1}\dot{x}^{k-1}\rangle.$$

The averaging of the last term can be carried out by using the Furutsu-Novikov formula,

$$k\langle\xi(t)x^{q-k+1}\dot{x}^{k-1}\rangle=-k(k-1)\sigma^2\langle x^{q-k+2}\dot{x}^{k-2}\rangle,$$

thus establishing a closed system for the evolution of the $q+1$ moments $\langle x^{q-k}\dot{x}^k\rangle$. This system of equations can be expressed with the help of a sparse matrix

$$\begin{pmatrix}
0 & q & 0 & 0 & & & & \\
-E & 0 & q-1 & 0 & \ddots & & & \\
2\sigma^2 & -2E & 0 & q-2 & \ddots & & & \\
0 & 2\times 3\sigma^2 & -3E & 0 & \ddots & & & \\
 & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & & 0 & 2 & 0 \\
 & & & & & (1-q)E & 0 & 1 \\
 & & & & & (q-1)q\sigma^2 & -qE & 0
\end{pmatrix}, \tag{5}$$

whose eigenvalue with the largest real part determines the exponential growth of the moments of order q. By definition (3) the generalized LE $L(q)$ for even q is thus equal to this eigenvalue divided by q (for odd $q\langle x^q\rangle\neq\langle|x^q|\rangle$ and this approach does not provide the LE).

In the simplest case of $q=2$ the generalized LE $L(2)$ is a solution of the cubic equation $\gamma^3+E\gamma-0.5\sigma^2=0$:

$$L(2)=\begin{cases} 2^{-2/3}\sigma^{2/3}\left[\left[1+\sqrt{1+\left(\frac{4^{2/3}E}{3\sigma^{4/3}}\right)^3}\right]^{1/3}-\frac{4^{2/3}E}{3\sigma^{4/3}\left[1+\sqrt{1+\left(\frac{4^{2/3}E}{3\sigma^{4/3}}\right)^3}\right]^{1/3}}\right] & \text{if} \quad \frac{E}{\sigma^{4/3}}\geqslant-\frac{3}{4^{2/3}} \\ 2\sqrt{\frac{|E|}{3}}\cos\left[\frac{1}{3}\arctan\sqrt{\left(\frac{4^{2/3}|E|}{3\sigma^{4/3}}\right)^3-1}\right] & \text{if} \quad \frac{E}{\sigma^{4/3}}<-\frac{3}{4^{2/3}}. \end{cases} \tag{6}$$

For larger q one has to find the roots of the corresponding polynomial of order $q+1$ numerically, but this is a straightforward task.

C. Multiscaling in terms of LEs

As follows from the analytic expressions for the generalized LEs above, they are in general different, which means multiscaling. Here we recall the relation between the generalized LEs and the fluctuations of the usual LE, which are described by the probability density $P(\lambda;t)$. Using Eq. (2) we can write $A(t)=\exp[t\lambda(t)]$, thus the cumulants $K_q(t)$ of the random process $\lambda(t)$ can be related to the moments of $A(t)$:

$$\langle A^q(t)\rangle=\langle\exp[q\lambda(t)t]\rangle=\exp\left[\sum_1^\infty\frac{q^nt^n}{n!}K_n(t)\right].$$

The first two cumulants, which correspond to the mean and the variance of $\lambda(t)$, respectively, scale for large times as follows:

$$K_1(t)=\langle\lambda(t)\rangle\underset{t\to\infty}{\rightarrow}\Lambda,\ K_2(t)=\langle[\lambda(t)-\Lambda]^2\rangle\underset{t\to\infty}{\rightarrow}D/t$$

with the diffusion constant D. The variance vanishes for large t in accordance with the self-averaging property of the local LE. Hence, by definition (3), the generalized LE is the asymptotic cumulant-generating function of $P(\lambda;t)$:

$$L(q)=\lim_{t\to\infty}\sum_1^\infty\frac{q^{n-1}t^{n-1}}{n!}K_n(t)$$

$$=\Lambda+\frac{q}{2}D+\lim_{t\to\infty}\sum_3^\infty\frac{q^{n-1}t^{n-1}}{n!}K_n(t). \tag{7}$$

This is just the Taylor expansion of $L(q)$ around $q=0$ with coefficients related to the cumulants of the local LE.

Now let us demonstrate a more direct connection between $L(q)$ and $P(\lambda;t)$. For $t\gg1$ the probability density $P(\lambda;t)$ of the local LE can be written in the scaling form $\exp[-tf(\lambda)]$ [9,14], where the entropy function $f(\lambda)$ is connected with the generalized LE via a Legendre transformation [2,10]:

$$f(\lambda)=q\lambda-qL(q),\quad \frac{d}{dq}qL(q)=\lambda. \tag{8}$$

The expansion of the entropy function around $\lambda=\Lambda$ is $f\approx(\lambda-\Lambda)^2/2D$, which yields a Gaussian distribution of the local LE. In the tails, however, deviations from the Gaussian will generally appear. In case the Gaussian approximation holds, higher cumulants $(n\geqslant3)$ vanish:

$$L(q)=\Lambda+\frac{Dq}{2}. \tag{9}$$

Then all generalized LEs are fully determined by two coefficients Λ and D [or, equivalently, by Λ and $L(2)$] and this situation can be characterized as "monoscaling." However, we will show below that for the noise-driven oscillator (1) this holds only for $|E|\gg\sigma^{4/3}$.

D. Parameter scaling

Before proceeding with the detailed analysis of the LEs, we explore the scaling dependence on the parameters E,σ. The analytical expressions (4) and (6) suggest the scaling relation $L(q)=\sigma^{2/3}f_q(E\sigma^{-4/3})$. To show that this scaling holds for all the exponents $L(q)$ we perform the time rescaling $t=(|E|/\sigma^2)\tau$ in Eq. (1), whereupon it can be written in the following form:

$$\ddot{x}+\left[\left(\frac{E}{\sigma^{4/3}}\right)^3+\left(\frac{|E|}{\sigma^{4/3}}\right)^{3/2}\eta(\tau)\right]x=0,$$

$$\langle\eta(\tau)\eta(\tau')\rangle=2\delta(\tau-\tau'). \tag{10}$$

The LEs determined by Eq. (10) obviously depend only on the parameter $\varepsilon\equiv E\sigma^{-4/3}$ as $\overline{L}(q,\varepsilon)$. Returning back to time t we have to reset the time scale by multiplying these exponents by $\sigma^2/|E|$; this gives for the LEs

$$L(q,E,\sigma)=\sigma^{2/3}\frac{\sigma^{4/3}}{|E|}\overline{L}(q,\varepsilon)\equiv\sigma^{2/3}\tilde{L}(q,\varepsilon). \tag{11}$$

The essential behavior is presented by the exponents $\tilde{L}(q,\varepsilon)$; thus throughout the rest of the paper this quantity is examined. For simplicity we will omit the tilde in the following;

relation to previous formulas can be achieved by inserting ε instead of $E\sigma^{-4/3}$ in the corresponding expressions (4) and (6).

III. GAUSSIAN SCALING FOR LARGE VALUES OF $|\varepsilon|$

In this section we demonstrate, using approximate methods, that for large $|\varepsilon|$ the Gaussian approximation to the distribution of the local LE or, equivalently, Eq. (9) holds.

A. Large positive values of ε

For positive ε a standard transformation to amplitude and phase variables can be made: $x = A\sin\psi$, $\dot{x} = \sqrt{\varepsilon}A\cos\psi$. The equations of motion then become

$$\dot{\psi}(t) = \sqrt{\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\xi(t)\sin^2\psi, \quad \dot{A}^q(t) = -\frac{q}{2\sqrt{\varepsilon}}\xi(t)A^q\sin 2\psi, \tag{12}$$

where the equation for the amplitude has been generalized to the equation for its qth power. With $u = \ln A$ the largest LE is given by

$$\Lambda = \langle \dot{u} \rangle = -\frac{1}{2\sqrt{\varepsilon}}\langle \xi(t)\sin 2\psi \rangle, \tag{13}$$

where the averaging is accomplished with the stationary distribution of ψ.

For large positive ε the deterministic phase velocity $\sqrt{\varepsilon}$ in Eq. (12) dominates over the typical diffusion rate $2/\varepsilon$. Thus for $\varepsilon^{3/2} \gg 1$ the probability density of the phase becomes uniform in the interval $[0,2\pi]$ and averaging the corresponding Fokker-Planck equation over ψ yields the evolution of the reduced probability density $\varrho(u,t)$ [1]:

$$\dot{\varrho}(u,t) = \left[-\frac{1}{4\varepsilon}\partial_u + \frac{1}{8\varepsilon}\partial_{uu} \right]\varrho(u,t).$$

This is equivalent to simple Brownian motion with a constant drift $1/4\varepsilon$. Hence u is normally distributed and for the exponents we have

$$\Lambda = \frac{1}{4\varepsilon}, \quad L(q) = \left(1 + \frac{q}{2}\right)\Lambda \rightarrow D = \Lambda. \tag{14}$$

The last statement is known as single parameter scaling, the distribution of the local LE being determined by its mean value alone. If one considers the parametric oscillator (1) as a continuous approximation to the discrete Anderson model for large wavelengths, i.e., around the lower band edge, then negative ε corresponds to the band gap of the Anderson model, the band edge being located at $\varepsilon = 0$. Increasing positive ε translates to an approach to the band center where single parameter scaling is known to exist. This property has been widely discussed in the context of Anderson localization [11–13].

B. Large negative values of ε

For negative ε one can transform to the eigenvectors of the noiseless system: $x = \tilde{x} + \tilde{y}$, $\dot{x} = |\varepsilon|^{1/2}(\tilde{x} - \tilde{y})$; whereupon the equations of motion become

$$\dot{\tilde{x}} = |\varepsilon|^{1/2}\tilde{x} + \frac{1}{2|\varepsilon|^{1/2}}\xi(t)(\tilde{x} + \tilde{y}),$$

$$\dot{\tilde{y}} = -|\varepsilon|^{1/2}\tilde{y} - \frac{1}{2|\varepsilon|^{1/2}}\xi(t)(\tilde{x} + \tilde{y}).$$

For large $|\varepsilon|^{3/2}$ the damping ($\sim |\varepsilon|^{1/2}$) of $\tilde{y}$ dominates over the fluctuations ($\sim 1/|\varepsilon|$); thus $\tilde{y}$ can be neglected compared to $\tilde{x}$ in the equation for $\dot{\tilde{x}}$. Hence the time evolution of $u = \ln|\tilde{x}|$ is given by

$$\dot{u} = |\varepsilon|^{1/2} + \frac{1}{2|\varepsilon|^{1/2}}\xi(t),$$

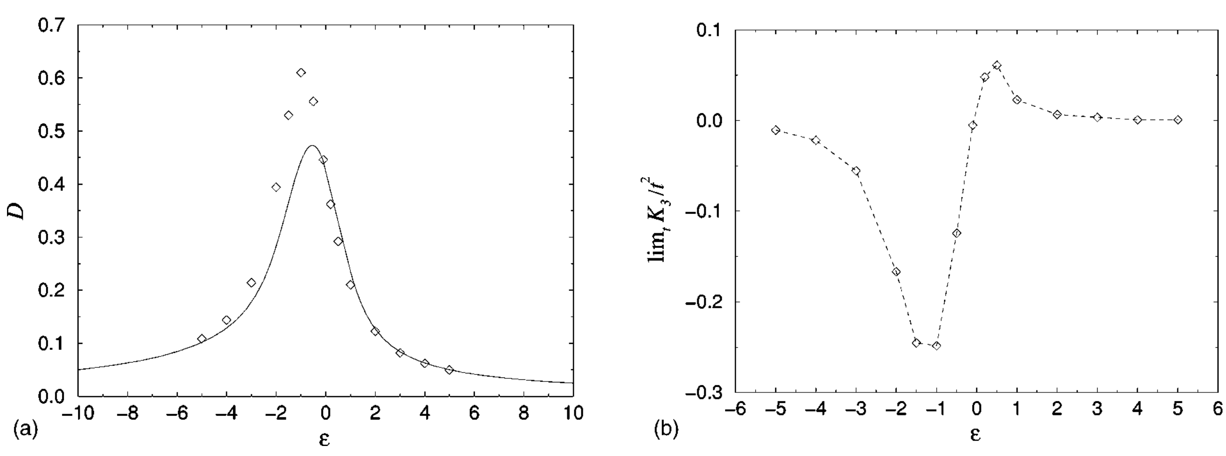

FIG. 1. (a) Diffusion constant D as a function of the frequency ε; $\sigma^2 = 1$. The diamonds show the numerical result whereas the solid line depicts the Gaussian presumption $D = L(2) - \Lambda$. (b) Numerical result for the limiting cumulant $\lim_{t\to\infty} K_3/t^2$; the dashed line is to improve readability.

which again leads to Gaussian distribution of the local LE, the latter given for large times by u/t. For the generalized Lyapunov exponents we obtain

$$\Lambda = |\varepsilon|^{1/2}, \quad L(q) = |\varepsilon|^{1/2} + \frac{q}{2|\varepsilon|} \approx |\varepsilon|^{1/2}. \tag{15}$$

Here there is no single parameter scaling (14) because $D = \Lambda^{-2}$. Notice that in both cases, $\varepsilon > 0$ and $\varepsilon < 0$, the asymptotic results are obtained for $|\varepsilon|^{3/2} \gg 1$.

IV. NON-GAUSSIAN FLUCTUATIONS

A. Parameter range of non-Gaussian fluctuations

We have demonstrated that for large $|\varepsilon|$ the distribution of local LEs is nearly Gaussian. Next, we would like to show that this does not hold for small $|\varepsilon|$ (see also [7]), with two numerical tests regarding the second and the third cumulant.

Suppose that the local LE is normally distributed. Then the whole set of generalized LEs can be expressed, according to Eq. (9), in terms of the analytically known exponents $\Lambda, L(2)$:

$$L(q) = \Lambda + \frac{D}{2} q = \Lambda + \frac{q}{2}[L(2) - \Lambda]. \tag{16}$$

In particular, the diffusion constant D equals $L(2) - \Lambda$ and the third cumulant K_3 in expansion (7) vanishes.

In Fig. 1(a) the numerically computed diffusion constant for the noise-driven oscillator is compared with the Gaussian assumption (16), indicating that there are deviations for values of ε close to zero. The coefficient $\lim_{t\to\infty} K_3/t^2$ of the cumulant expansion (7) is plotted in Fig. 1(b). It is clearly different from zero for small $|\varepsilon|$.

These results suggest that in the intermediate regime, $|\varepsilon| \lesssim 1$, the linear form of $L(q)$ is not correct. This is elucidated by writing the local LE explicitly in terms of amplitude and phase [compare Eq. (13)]:

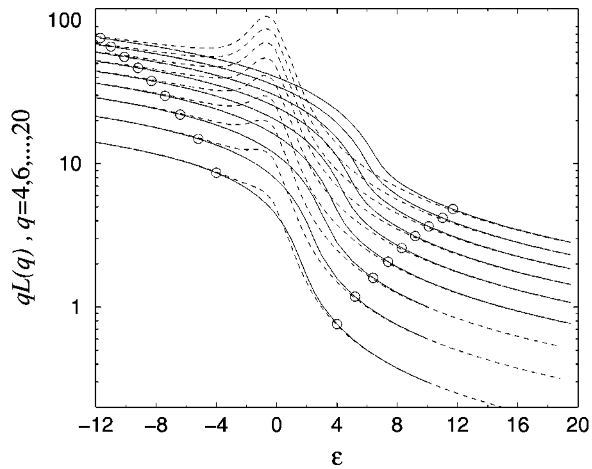

FIG. 2. $qL(q)$ vs ε for values of q increasing from $q=4$ at the bottom to $q=20$ at the top. The solid lines show the numerical result, the dashed lines the parabolic approximation (16). The threshold $q/|\varepsilon|^{3/2} = 0.4$ is marked by the circles.

$$\lambda(t) = -\frac{1}{2\sqrt{\varepsilon}t} \int_0^t \xi(\tau) \sin 2\psi(\tau) d\tau. \tag{17}$$

Because the noise and the phase are coupled according to Eq. (12), the process $\xi(\tau)\sin 2\psi(\tau)$ resembles Gaussian white noise only for $\varepsilon^{3/2} \gg 1$, as was shown in the previous section. In terms of the localization theory the point $\varepsilon = 0$ corresponds to the band edge, where indeed a complicated behavior of the corresponding distributions is expected [11–13].

Taking into account that the limiting results (14) and (15) are obtained for $|\varepsilon|^{3/2} \gg 1$, we assume hypothetically that the relevant scale variable is given by $|\varepsilon|^{3/2}$ and hence the cumulants of the density $P(\lambda;t)$ vanish for large $|\varepsilon|$ as

$$K_n \sim \left(\frac{1}{|\varepsilon|^{3/2}} \right)^n \quad \text{for} \quad n \geq 3. \tag{18}$$

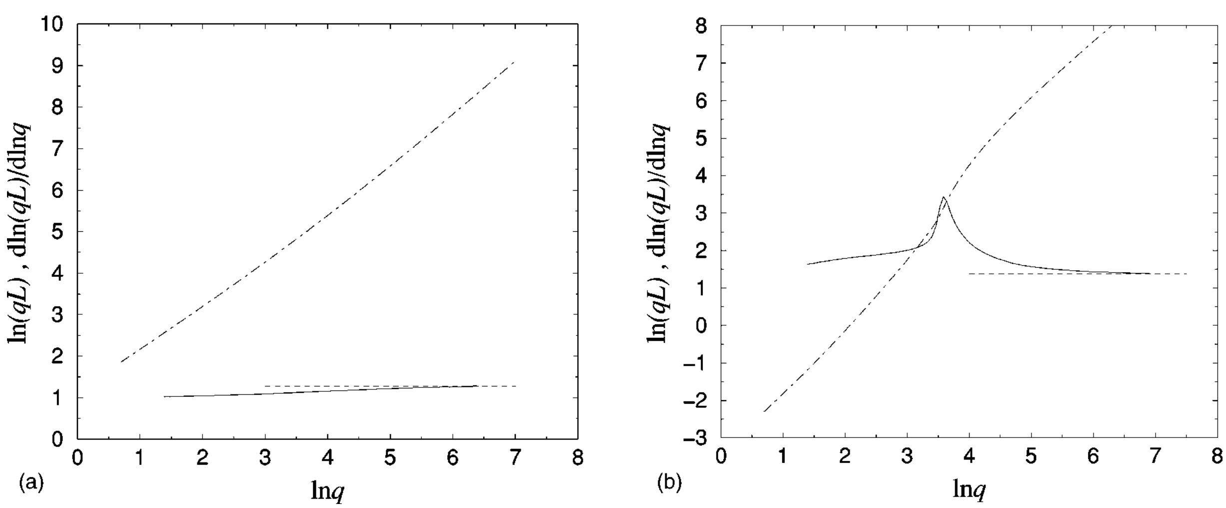

FIG. 3. $\ln[qL(q)]$ vs $\ln q$ (dot-dashed) for $\varepsilon = -10$ (a) and $\varepsilon = 10$ (b). The solid lines are numerical derivatives; $d\ln[qL(q)]/d\ln q$; the dashed lines correspond to $\alpha = 1.28$ and $\alpha = 1.38$, respectively.

Then the expansion (7) is a power series in the parameter $q/|\varepsilon|^{3/2}$. According to this assumption, the generalized LE $L(q)$ deviates from the Gaussian value (16) when this parameter is large.

To test this numerically, we show in Fig. 2 the exact generalized LEs together with the Gaussian approximation (16), as a function of ε for several values of the parameter q. The indicated threshold $q/|\varepsilon|^{3/2}=0.4$ well divides the region of agreement between the correct value and the approximation from the region where these values strongly disagree, thus confirming our hypothesis.

B. Asymptotic scaling of generalized LEs

In this section we study the asymptotic behavior of generalized LEs for large q. This problem can be formulated as the problem of asymptotic properties of the eigenvalues of the matrix (5) as $q\to\infty$. We expect the scaling to be a power law

$$L(q)\sim q^{\alpha-1}, \quad \alpha(\varepsilon)\in(1,2). \tag{19}$$

Because the largest element of the matrix (5) scales as q^2 for large $q, L(q)\sim q$ sets an upper limit to the scaling (19). The numerical results presented in Fig. 3 for two values of ε give $\alpha=1.28$ for $\varepsilon=-10$ and $\alpha=1.38$ for $\varepsilon=10$.

The fact that asymptotically $\alpha<2$ means that the tails of the distribution of the local LE are suppressed in comparison to the Gaussian form. Indeed, by virtue of the Legendre transformation (8), the scaling of $L(q)$ for large q translates into a scaling of the entropy function $f(\lambda)$ for $\lambda\gg1$:

$$f(\lambda)\sim(\alpha-1)\left(\frac{\lambda}{\alpha}\right)^{\alpha/(\alpha-1)} \quad \text{for} \quad \lambda\gg1. \tag{20}$$

The linear form (16) would give $\alpha=2$, i.e., a Gaussian form of $P(\lambda;t)$. For $\alpha<2$, however, $f(\lambda)$ obeys a power law with an exponent $\alpha/(\alpha-1)>2$; i.e., $P(\lambda;t)$ decays faster than the Gaussian distribution for large values of λ.

We note also a definite crossover in the scaling in Fig. 3(b), which is clearly seen as a maximum in the dependence of the slope $d\ln qL/d\ln q$ on q. The position of this crossover, q_c, is plotted as a function of ε in Fig. 4 which supports a scaling $q_c\sim\varepsilon^{3/2}$. This is further support for the scaling relation (18) separating Gaussian and non-Gaussian behavior of LEs.

We emphasize that convergence problems of the numerical methods, used to solve the eigenvalue problem stated by Eq. (5), did not allow us to find the asymptotic exponent α with sufficient accuracy. This is due to the growing size of the matrix ($\sim q$) and the strong difference in magnitude of its elements ($\sim q^2$) for large values of q.

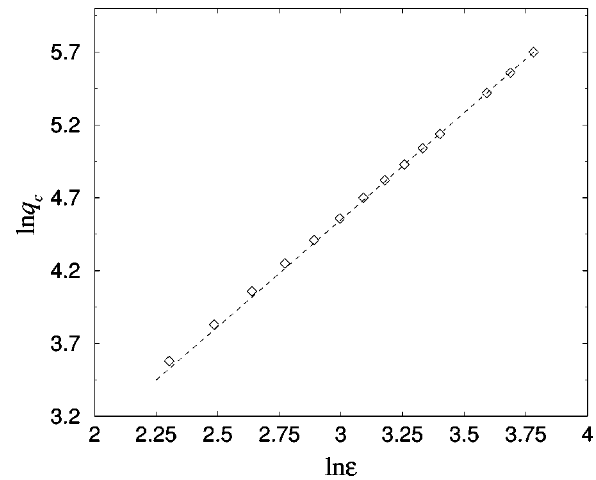

FIG. 4. The crossover $\ln q_c$ vs $\ln\varepsilon$. The dashed line has slope $1.47\approx3/2$.

V. CONCLUSION

We presented numerical and analytical arguments confirming a nontrivial distribution of the local Lyapunov exponent in the case of the linear noise-driven oscillator (1). In order to describe multiscaling, we considered the generalized Lyapunov exponents $L(q)$, which characterize the fluctuations of the local LE. With the help of a scaling relation we were able to represent all the exponents $L(q)$ as functions of a renormalized "energy" ε. A linear form of the generalized LEs is equivalent to a normal distribution of the local Lyapunov exponent which, however, is valid for the noise-driven oscillator only in the limit $\varepsilon\to\infty$. To be more precise, the normal distribution is only an approximation in the vicinity of the mean value. We have found that the parameter range where the linear approximation for the exponent $L(q)$ is valid depends on the index q and reads $|\varepsilon|^{3/2}\gg q$. In other words, the exponential growth of moments $\langle A^q\rangle$ of qth order is determined by the Gaussian part of the distribution within this parameter range. Our numerical findings in the limit $q\gg1$ suggest a scaling relation of $L(q)\sim q^{\alpha-1}$. The corresponding exponent $\alpha\approx1.4$ describes the probability of large deviations of the local LE from its average value via the Legendre transformation.

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